

Bifurcation scenario to Nikolaevskii turbulence in small systems

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Abstract

We show that the chaos in Kuramoto-Sivashinsky equation occurs through period-doubling cascade (Feigenbaum scenario), in contrast, the chaos in Nikolaevskii equation occurs through torus-doubling bifurcation (Ruelle-Takens-Newhouse scenario).

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We concern the onset of spatiotemporal chaos exhibited by Nikolaevskii equation [1, 2] in one-dimensional space with periodic boundary conditions,

$$\partial_t \psi(x, t) = -\partial_x^2 [\epsilon - (1 + \partial_x^2)^2] \psi - (\partial_x \psi)^2. \quad (1)$$

Because this class of spatiotemporal chaos appears in many types of physical systems, studying Eq. (??) is important (see, e.g., the introduction part of [5]). When we introduce a variable v defined as $v \equiv 2\partial_x \psi$, we can rewrite Eq. (1):

$$\partial_t v(x, t) = -\partial_x^2 [\epsilon - (1 + \partial_x^2)^2] v - v \partial_x v, \quad (2)$$

which we use in the following. This equation has the two parameters, the bifurcation parameter ϵ and the system size L . We know already that the spatiotemporal chaos in Eq. (2) occurs supercritically when ϵ increases in sufficiently large system [3]. This can be confirmed from the fact that the amplitudes of turbulent fluctuations vanish at the limit $\epsilon = 0$ [4, 5]. In this paper, we concentrate how the chaos occurs when the system size increases from sufficiently small size, i.e., we elucidate bifurcation route to chaos in Eq. (2) increasing the aspect ratio. If characteristic wavelengths of spatiotemporal patterns are comparable to the system size, then the number of active modes accommodated by the system is of order 1, i.e., the systems behave in much the same way as systems of a few degrees of freedom. Thus, we can conjecture that the bifurcation route to chaos in Eq. (2) when the aspect ratio increases is period-doubling cascade (Feigenbaum scenario) or intermittency transition (Pomeau-Mannville scenario) or breakdown of T^2 -torus (Ruelle-Takens-Newhouse scenario) [6, 7, 8].

First, we concern, instead of Eq. (2), a well-known model exhibiting spatiotemporal chaos, Kuramoto-Sivashinsky (KS) equation

$$\partial_t v(x, t) = -\partial_x^2 (1 + \partial_x^2) v - v \partial_x v. \quad (3)$$

Its trivial solution $v = 0$ is unstable with respect to perturbations having wavenumbers $q \in (0, 1)$ and stable with respect to perturbations having the other wavenumbers. Since the wavenumbers are, in finite-size space L , restricted to being $2\pi\nu/L$ with any integer ν , all wavenumbers lies out of the unstable band $(0, 1)$ if L is smaller than $L_c^{KS} \equiv 2\pi$, and then the uniform steady state $v = 0$ is stable. When we increase L slightly from L_c^{KS} , a few modes enter in the band $(0, 1)$, i.e., they destabilize the uniform steady state, and the

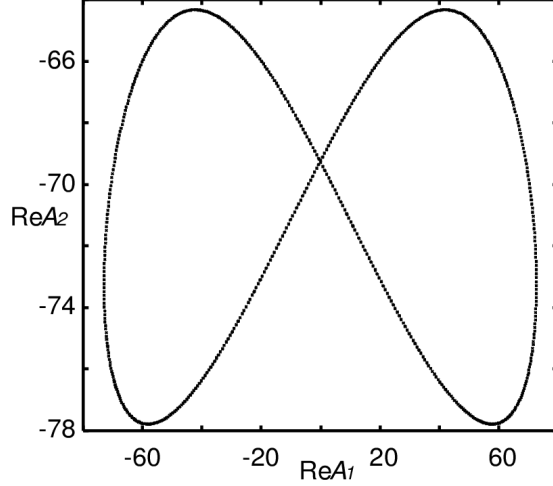


FIG. 1: Periodic attractor obtained from the KS equation with $L = 2.8L_c^{KS}$.

system exhibits a dynamical state that must be described as a low-dimensional attractor. To visualize the attractor, we chose a subspace $\text{Re}A_1 - \text{Re}A_2$, onto which we project the attractor, where $A_\nu(t)$ are the Fourier coefficients defined as

$$v(x, t) = \sum_{\nu \in \mathbb{Z}} A_\nu(t) e^{i2\pi\nu x/L}. \quad (4)$$

This choice of subspace is relatively good to see topology of the attractor. When we increase L from L_c^{KS} , the trajectory $(\text{Re}A_1(t), \text{Re}A_2(t))$ is periodic at first as shown in Fig. 1. Then, the attractor loses its symmetry as shown in Fig. 2. After that, the period of the trajectory is doubled as shown in Fig. 3. Finally, the attractor becomes chaotic as shown in Fig. 4. The return map on a Poncaré section of this chaotic attractor is unimodal, as shown in Fig. 5, which means that the KS chaos occurs through the period-doubling cascade (Feigenbaum scenario). In Ref.[9], Y. Kuramoto shows a similar unimodal map for Eq. (3) with another boundary conditions $v = \partial_x^2 v = 0$.

Now, we elucidate the onset of chaos in Eq. (2). This equation also has the trivial solution $v = 0$. Its unstable band is $(\sqrt{1 - \sqrt{\epsilon}}, \sqrt{1 + \sqrt{\epsilon}})$. Thus, the critical system size is $L_c^N \equiv 2\pi/\sqrt{1 + \sqrt{\epsilon}}$ (, which is equal to $5.4766 \dots$ when $\epsilon = 0.1$). In the following, we use $\epsilon = 0.1$. This value is sufficiently small in order to observe characteristic spatiotemporal chaos exhibited by Eq. (2) in large systems [5]. Using Eq. (4), we choose a subspace $\text{Re}A_1 - \text{Im}A_1$, which is relatively good choice to see topology of the attractor. The attractor in slightly larger system-size than L_c^N is shown in Fig. 6, where the trajectory seems to be quasi-

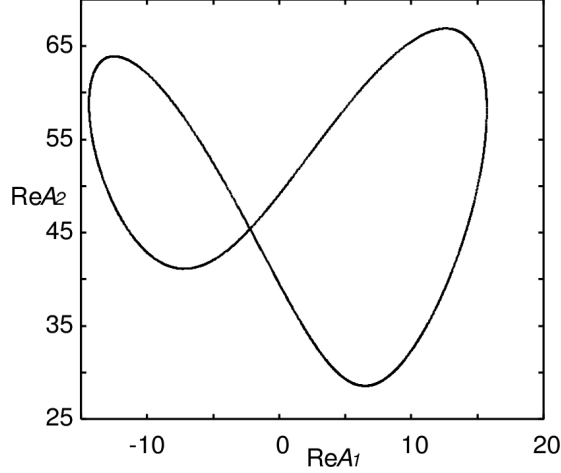


FIG. 2: Symmetry-broken attractor obtained from the KS equation with $L = 2.87L_c^{KS}$.

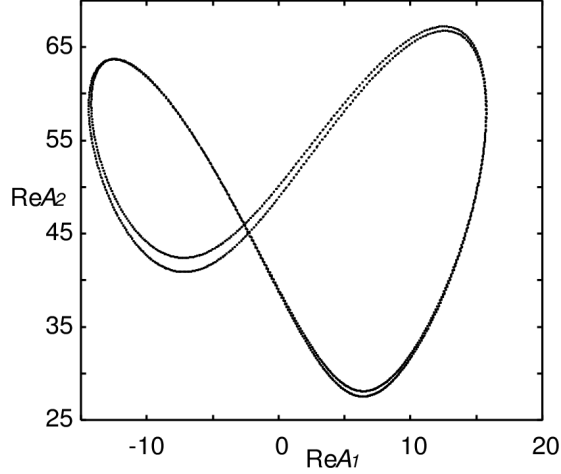


FIG. 3: Period-Doubled attractor obtained from the KS equation with $L = 2.871L_c^{KS}$.

periodic. In fact, the return map on a Poncaré section of this trajectory is a closed orbit as shown in Fig. 7, which means that the attractor is T^2 torus. When we still increase L , we see the torus doubled first, as shown in Fig. 8, and then chaos, as shown in Fig. 9. Thus, the Nikolaevskii chaos occurs through the breakdown of T^2 -torus (Ruelle-Takens-Newhouse scenario).

Finally, we comment a future issue. In large systems, the spatial power spectrum of the Nikolaevskii equation with $\epsilon \geq O(0.1)$ is qualitatively indistinguishable from that of the Kuramoto-Sivashinsky equation [5]. According to this fact, does the Nikolaevskii equation with $\epsilon \geq O(0.1)$ exhibit the Feigenbaum scenario as the Kuramoto-Sivashinsky equation ?

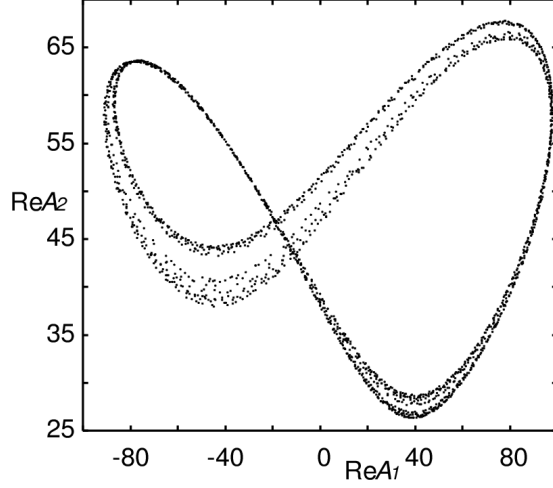


FIG. 4: Chaotic attractor obtained from the KS equation with $L = 2.872L_c^{KS}$.

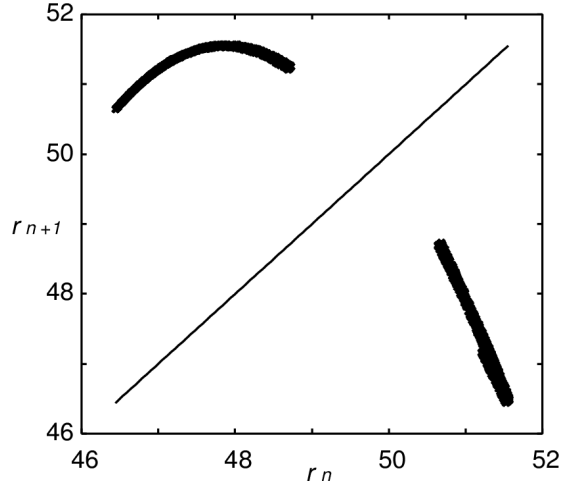


FIG. 5: Return map for the chaotic trajectory shown in Fig. 4. That trajectory crosses the Poncaré section $\text{Re}A_1 = 0$ near $\text{Re}A_2 = 40$ and near $\text{Re}A_2 = 50$ one after another. r_n are defined as $\text{Re}A_2$ at the intersections near $\text{Re}A_2 = 50$. The KS chaos occurs through the period-doubling cascade (Feigenbaum scenario) because this map is unimodal.

If so, how does the Ruelle-Takens-Newhouse scenario change to the Feigenbaum scenario ?

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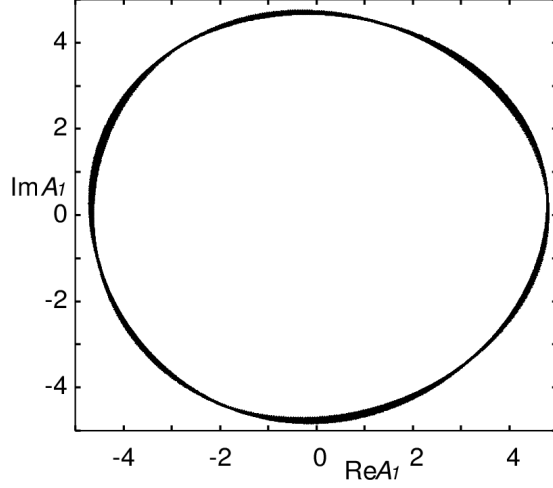


FIG. 6: Quasi-periodic attractor obtained from the Nikolaevskii equation with $L = 3.97L_c^N$ and $\epsilon = 0.1$. In this figure, the width of the orbit undulates slightly. However, we confirmed that this undulation disappears when we plot the trajectory for a longer time interval. It means that the trajectory has two incommensurate frequencies.

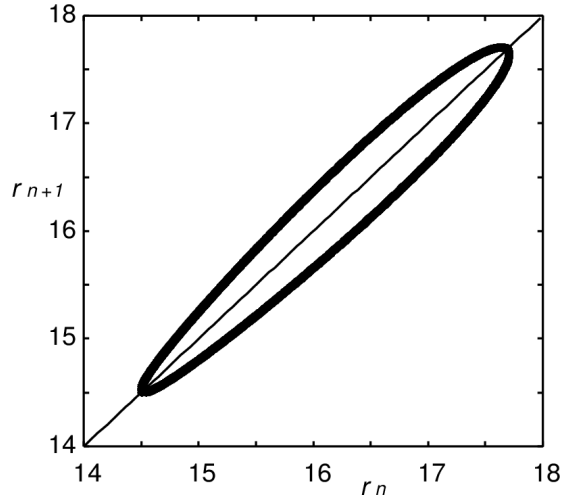


FIG. 7: Return map obtained from the Nikolaevskii equation with $L = 3.98L_c^N$ and $\epsilon = 0.1$. r_n are defined as $|A_1|$ on the Poncaré section $\text{Re}A_1 = \text{Im}A_1 > 0$. The closed orbit is a cross section of T^2 torus.

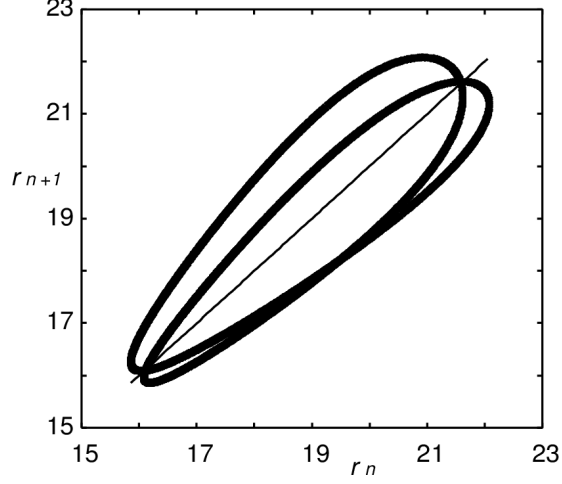


FIG. 8: Return map obtained from the Nikolaevskii equation with $L = 3.9845L_c^N$ and $\epsilon = 0.1$. The definition of r_n is the same as in Fig. 7. We see T²-torus-doubling.

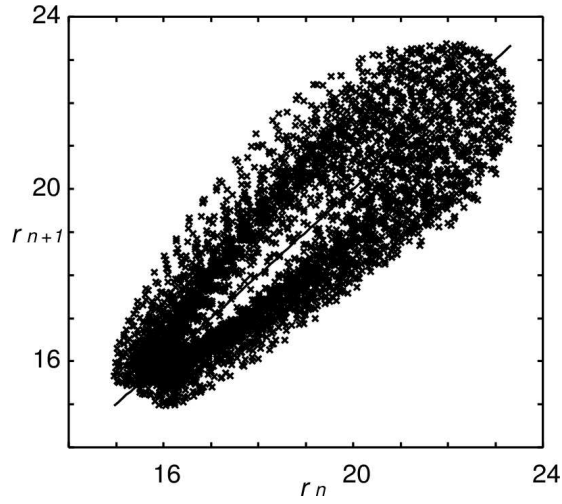


FIG. 9: Return map obtained from the Nikolaevskii equation with $L = 3.985L_c^N$ and $\epsilon = 0.1$. The definition of r_n is the same as in Fig. 7. The Nikolaevskii chaos occurs through the T²-torus-doubling bifurcation (Ruelle-Takens-Newhouse scenario).

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